# Appendix D
Derivation of Tapered Beam Stiffness Matrices

In ADAMS/WT, we create a tapered beam effect using the ADAMS FIELD element, which allows us to directly input the terms in the stiffness matrix. To derive the FIELD matrix terms which correspond to a linearly tapered beam, we go to the finite element method. Consider the transverse deflection of a single tapered beam element, shown below with the local coordinate system and the nodal displacements. Axial and torsional response will be considered later.
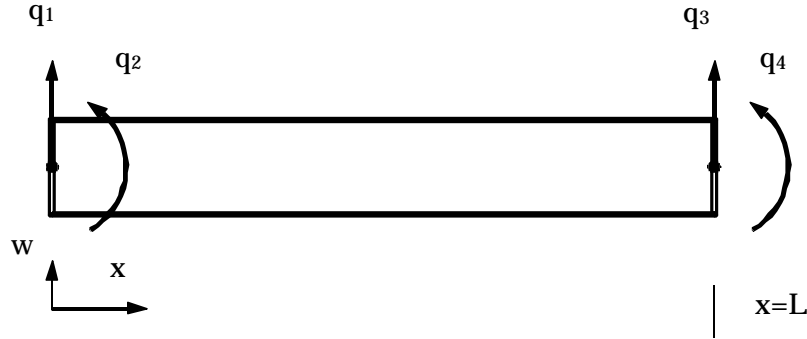
As is standard practice in finite element analysis (for example, see Bathe[1i]), we assume that the deflected shape can be approximated by a cubic polynomial for which the nodal displacements are the boundary conditions. In order to determine the element deflected shape from the nodal displacements, we assume that

$$w(x) = \sum_{1}^{4} H_i(x) q_i$$

A little bit of algebra then gives us the definitions for the shape functions, H_i :

$$H1(x, L) := 2 \cdot \frac{x^3}{L^3} - 3 \cdot \frac{x^2}{L^2} + 1$$

$$H2(x, L) := L \cdot \left(\frac{x^3}{L^3} - 2 \cdot \frac{x^2}{L^2} + \frac{x}{L} \right)$$

$$H3(x, L) := -2 \cdot \frac{x^3}{L^3} + 3 \cdot \frac{x^2}{L^2}$$

$$H4(x, L) := L \cdot \left(\frac{x^3}{L^3} - \frac{x^2}{L^2} \right)$$

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[1] Bathe, K.-J., _Finite Element Procedures in Engineering Analysis_, Prentice-Hall, NY, 1982

Returning to the finite element method, we find the definition of the terms of the stiffness matrix is:

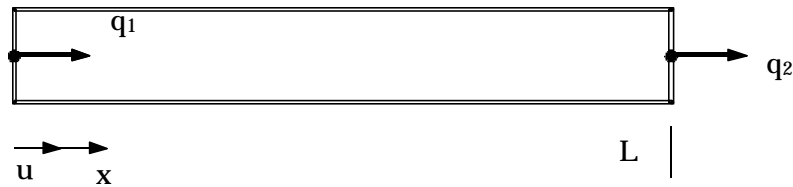$$k_{ij} = \int_0^L EI \frac{d^2 H_i}{dx^2} \frac{d^2 H_j}{dx^2} dx$$

We then assume that the variation in EI is linear over the length of the element, and that we know the values at the end points.

$$EI(x, L) = EI_o + \left(\frac{EI_L - EI_o}{L} \right) x$$

We can then complete the integrals symbolically to get only those terms which are needed for the FIELD statement. These are

$$k_{33} = \frac{6}{L^3} \left(EI_L + EI_o \right)$$

$$k_{34} = \frac{-2}{L^2} \left(2 EI_L + EI_o \right)$$

$$k_{44} = \frac{1}{L} \left(3 EI_L + EI_o \right)$$

For the axial and torsional degrees of freedom, we follow the same methodology, but instead of a cubic assumed displacement over the length of the element, we assume a linear function. Again, this is standard practice, both for finite element analysis and for the ADAMS BEAM. Looking at the axial extension (u) problem (torsion is exactly analogous), we find

Similarly to the bending case, we expand the deflection in terms of the shape functions:

$$u(x) = \sum_1^4 H_i(x) q_i$$

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where the shape functions in this case are

$$H1\,(\,x,\,L\,)\;:=1-\frac{x}{L}$$

$$H2\,(\,x,\,L\,)\;:=\frac{x}{L}$$

Again returning to the finite element method, we find the definition of these terms in the stiffness matrix is:

$$k_{ij}=\int_{0}^{L}EA\frac{dH_{i}}{dx}\frac{dH_{j}}{dx}dx$$

We then assume that the variation in EA is linear over the length of the element, and that we know the values at the end points.

$$EA(x,L)=EA_{o}+\left(\frac{EA_{L}-EA_{o}}{L}\right)x$$

We can then complete the integral symbolically to get the term which is needed for the FIELD statement. This is

$$k_{11}=\frac{1}{2L}\left(EA_{L}+EA_{o}\right)$$

For the torsion degree of freedom, just substitute GJ for EA.

$$k_{44}=\frac{1}{2L}\left(GJ_{L}+GJ_{o}\right)$$

In order to create the correct FIELD matrix, it is important to remember that for each FIELD element, ADAMS computes the forces on the I marker due to displacements of the I marker *in the J marker's reference frame.* This is different from finite element methods, where the assumption of globally small displacements allows everything to be assembled in one base frame. Moreover, because ADAMS' formulation uses the displacement *difference* between the markers, only those terms listed should be incorporated into the matrix. It is also important to note that this linearly tapered (in stiffness, not cross section) beam is assumed to be straight in the unloaded configuration.

Note that the input to this element are the combined stiffness properties at each end of the beam, not the separate material and cross section properties, and that it is the combined properties which are assumed to vary linearly of the element length. That is, you must specify EI_y, EI_z, EA, GJ at the element boundaries, where these values are the running properties (*per unit length*) of in-plane and out-plane bending stiffness, extensional stiffness and torsional

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stiffness. As with all ADAMS force elements, the inertial properties associated with the beam
go separately into ADAMS PARTs (see Appendix E - Tapered Part Derivation).

To maintain consistency with the approach used for the ADAMS BEAM element, the coordinate
axes are set so that the blade runs along the positive x-axis from root to tip, with the z-axis
toward the leading edge and the y-axis in the positive flap direction (toward the suction side of
the airfoil, as shown below.
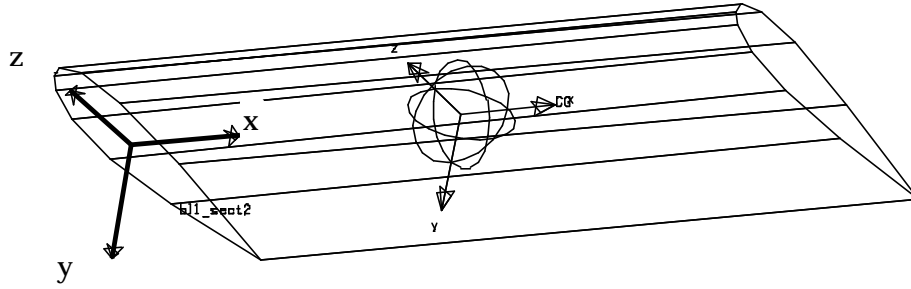
In the FIELD matrix, the rows correspond to x, y and z displacements, followed by rotations
about the x, y, and z axes. For convenience, all terms are listed here in the same order which
they would go into the FIELD statement (column-major order).

$$k_{11} = \frac{1}{2L}\left(EA_L + EA_o\right)$$

$$k_{21} = k_{31} = k_{41} = k_{51} = k_{61} = 0$$

$$k_{12} = 0$$

$$k_{22} = \frac{6}{L^3}\left(EI_{ZL} + EI_{Zo}\right)$$

$$k_{32} = k_{42} = k_{52} = 0$$

$$k_{62} = \frac{-2}{L^2}\left(2EI_{ZL} + EI_{Zo}\right)$$

$$k_{13} = k_{23} = 0$$

$$k_{33} = \frac{6}{L^3}\left(EI_{YL} + EI_{Yo}\right)$$

$$k_{43} = 0$$

$$k_{53} = \frac{2}{L^2}\left(2EI_{YL} + EI_{Yo}\right)$$

$$k_{63} = 0$$

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$$k_{14} = k_{24} = k_{34} = 0$$

$$k_{44} = \frac{1}{2L}\left(GJ_L + GJ_o\right)$$

$$k_{54} = k_{64} = 0$$

$$k_{15} = k_{25} = 0$$

$$k_{35} = \frac{2}{L^2}\left(2EI_{YL} + EI_{Yo}\right)$$

$$k_{45} = 0$$

$$k_{55} = \frac{1}{L}\left(3EI_{YL} + EI_{Yo}\right)$$

$$k_{65} = 0$$

$$k_{16} = 0$$

$$k_{26} = \frac{-2}{L^2}\left(2EI_{ZL} + EI_{Zo}\right)$$

$$k_{36} = k_{46} = k_{56} = 0$$

$$k_{66} = \frac{1}{L}\left(3EI_{ZL} + EI_{Zo}\right)$$